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Turns for the Lorentz group

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Abstract. A geometrical description of the composition law for pure inertial transformations in relativity is developed, in close analogy with the description of rotations by means of turns. This affords a very intuitive and visual image of all problems involving a composition of velocities, and clearly shows the appearance of hyperbolic geometry in relativistic kinematics. The pertinent geometrical constructions as well as some applications are discussed. For the case of light the preceding constructions are surprisingly simple, and from them emerges a connection between the geometrical idea of parallelism angle and the physical Doppler effect. A slide rule which can be used to carry out all needed computations is also described.

1. Introduction

Almost a century and a half ago, Hamilton developed a representation of the law of composition of rotations by means of quaternions (Hamilton 1853). An idea very closely related to Hamilton's treatment is that of turns. The turn associated with a rotation $R_{n\varphi}$, of axis $\boldsymbol{n}$ and angle φ, is a directed arc of length $\frac{1}{2}\varphi$ on the great circle orthogonal to $\boldsymbol{n}$ on the unit sphere. By means of these objects, the product of rotations is described through a 'parallelogram-like law', i.e. if these turns are translated on the great circles, until the head of the arc of the first rotation coincides with the tail of the arc of the second one, then the turn between the free tail and the head is associated to the product rotation. In recent times these objects have received some attention because they afford a very good way to give an intuitive understanding of some of the peculiar properties of rotations (Biedenharn and Van Dam 1965, Harter and dos Santos 1978, Hartung 1979, and especially Biedenharn and Louck 1981).

Although the strict attribution of turns to Hamilton appears to be inaccurate, the turns are so closely related to unimodular quaternions that, both for the sake of brevity and for distinguishing them from other similar objects to be introduced in the paper, we shall call them Hamilton's turns. In fact, Hamilton's turns are analogous, for spherical geometry, to the sliding vectors in Euclidean geometry: both have a parallelogram law of addition, which gives a quick method of representation and visualisation to the composition law of isometries in the geometry under consideration.

The purpose of the present article is to develop a similar method, which can be used for an easy visualisation of the composition of pure inertial transformations in special relativity. As is well known, this product is not, in general, a pure inertial transformation, but its product with a rotation. This physical property of the geometry

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of space–time causes e.g. the Thomas precession. The problem is studied in almost every book on relativity, directly by Moller (1972), by means of the spinorial representation of $\mathscr{L}$, or even using a purposeful procedure (Farach *et al* 1979). Whatever the method used for its derivation, the formulae giving the velocity $\boldsymbol{W}$ and the rotation R, arising from the 'composition' of velocities $\boldsymbol{V}$ and $\boldsymbol{V}'$, are relatively complicated. We extended in this paper the turns idea, and as a result we obtain a very appealing geometrical construction of $\boldsymbol{W}, R$ in terms of $\boldsymbol{V}$, $\boldsymbol{V}'$, which, although not intended as a substitute of analytical relationships, gives a quick visual image of the main features of this phenomenon.

In § 2 we briefly recall how hyperbolic geometry appears naturally in relativistic kinematics as the geometry of uniform motions. In § 3 we generalise the idea of turns, and develop a geometric procedure which gives good visualisation of the peculiarities involved in the relativistic composition of velocities. Section 4 is devoted to some examples. Finally, § 5 deals with the case of light, and a very simple geometrical construction for light aberration is obtained, and also an intriguing connection between the Doppler effect and the geometrical concept of angle of parallelism. A slide rule, which allows the practical development of the geometrical composition of pure inertial transformation, is given in the appendix.

2. Uniform motion geometries

There are many geometrical systems entangled in the four-dimensional kinematics described by some kinematical group G. We are restricting ourselves to absolute space–time theories (Anderson 1967). In fact each subgroup H of G can be considered as acting on each of the homogeneous spaces of H, and in some cases the corresponding actions coincide with those of the classical Cayley–Klein geometries. In this way, Euclidean geometry is physically realised as the space geometry in inertial frames, both in 'classical' and 'relativistic' physics (subgroup generated by $\langle \boldsymbol{P}, \boldsymbol{J}\rangle$, isomorphic in both cases to E(3), homogeneous space E(3)/SO(3) isometric to the Euclidean space). The event geometry in relativity is similarly linked to the Minkowski space–time $\mathscr{P}/\mathscr{L}$, whereas their classical counterpart, the so-called Galilean geometry, is less known, but beautifully discussed by Yaglom (1979).

But there is, at last, a third geometry hidden in kinematics. From the group theoretical viewpoint, it arises through the subgroup generated by $\langle \boldsymbol{K}, \boldsymbol{J}\rangle$ and its homogeneous space by the subgroup SO(3) of the rotations. In the Galilean case, this subgroup is isomorphic to E(3) (as a consequence of $[\boldsymbol{K}, \boldsymbol{K}]=0$), so that this geometry is also Euclidean, a fact implicitly recognised, e.g. when one uses the parallelogram law for the composition of velocities. But, for the relativistic case, as here $[\boldsymbol{K}, \boldsymbol{K}]=-\boldsymbol{J}$, this group is isomorphic to H(3), the hyperbolic group, and the corresponding geometry is hyperbolic.

Ordinary space geometry deals with the properties of the space formed by points which are obtained from one of them by all translations, at a fixed time and at a fixed state of motion (H and $\boldsymbol{K}$ having in some sense been 'neglected'). Similarly this unusual geometry deals *ab initio* with the properties of space whose points are the motions generated by $\boldsymbol{K}$ at a fixed space–time point. But the homogeneity of space–time in relativity, leading to the linear action of pure inertial transformations on the space–time, allows us to extend the *a priori* restricted meaning of that geometry, from 'uniform motions at fixed space–time point' to simply 'uniform motions', identifying, in passing, all uniform motions with the same velocity through different space–time points; each class being characterised, relative to a fixed one, by its velocity $\boldsymbol{V}(|\boldsymbol{V}|<1)$ or by its rapidity $\boldsymbol{\chi}$

$$\boldsymbol{\chi}=\frac{\boldsymbol{V}}{|\boldsymbol{V}|}\tanh^{-1}|\boldsymbol{V}|.$$

Then this 'pure inertial' geometry is but the so-called uniform motion geometry (Yaglom 1979) or geometry in the space of velocities (Fock 1959). Of course the fact that geometry is hyperbolic has been known from the early times of relativity (see e.g. Pauli 1958), but it has never deserved the popularity of the event Minkowskian geometry.

One may well ask for a more 'down-to-earth' description of this geometry. This can be stated using Minkowski geometry. To simplify matters we restrict ourselves here to the case of two space dimensions. A uniform motion including the event 0 is a straight time-like line passing through 0. Any such line, γ, meets the upper sheet $\Omega^+_{r=1}$ of the 'sphere' $\Omega_{r=1}$ only once

$$x^2+y^2-t^2=1 \qquad t>0$$

so that the set of uniform motions can be identified with the points of $\Omega^+_{r=1}$. Then a good measure of the 'deflection' between two such points P, P' is, of course, the relative rapidity of the corresponding motions, which is simply the Minkowskian angle between the lines γ, γ', given in turn by the Minkowski length of the arc of circle on the unit sphere $\Omega^+_{r=1}$ of centre 0 and between P and P'. So the plane of uniform motions has been isometrically identified with the upper sheet of the sphere $\Omega_{r=1}$; this with the induced Minkowski metric. It is a very simple exercise to show that $\Omega^+_{r=1}$, with this metric, is a surface with constant negative Gaussian curvature resulting in a hyperbolic plane.

Returning now to the three-dimensional case, we notice that the result of the preceding discussion is clear, because of the fact the Lie algebra of the Lorentz group

$$[J_i, J_j]=\varepsilon_{ijk}J_k \qquad [J_i, K_j]=\varepsilon_{ijk}K_k \qquad [K_i, K_j]=-\varepsilon_{ijk}J_k$$

is immediately seen to be isomorphic to the Lie algebra of the hyperbolic space group, spanned by P_i, L_i, $i=1,2,3$; P_i generating translations along three mutually orthogonal lines, and L_i generating rotations around the point of intersection of these lines. If the units of length and angle are the natural ones in hyperbolic geometry, then the isomorphism is simply

$$J_i\leftrightarrow L_i \qquad K_i\leftrightarrow P_i. \tag{1}$$

This isomorphism extends to the groups because their connectivity properties are the same. If the elements of the Lorentz group are parametrised by normal coordinates $\boldsymbol{\chi}$, $\boldsymbol{\varphi}$, in such a way that $V_{\boldsymbol{\chi}}$ is a pure inertial transformation of rapidity $\chi=|\boldsymbol{\chi}|$ in the space direction specified by the unit vector $\boldsymbol{m}$, $(\boldsymbol{\chi}=\chi\boldsymbol{m})$, and $R_{\boldsymbol{\varphi}}$ a spatial rotation of angle $|\boldsymbol{\varphi}|$ around the axis $\boldsymbol{n}$, $(\boldsymbol{\varphi}=\varphi\boldsymbol{n})$, we have

$$(V_{\boldsymbol{m},\chi}, R_{\boldsymbol{n},\varphi})=\exp(\chi\boldsymbol{mK})\exp(\varphi\boldsymbol{nJ}). \tag{2}$$

The isomorphism (1) gives the isometry M of the hyperbolic plane

$$M=\exp(\chi\boldsymbol{mP})\exp(\varphi\boldsymbol{nL})$$

as the image to (2). The Lorentz rotations are mapped into rotations around some point 0 in hyperbolic space, and pure inertial transformations go into translations along some line through 0, a line which may be labelled by $\boldsymbol{m}$.

Now take a product of two pure inertial transformations, say $V_{\boldsymbol{m}\chi}$ and $V_{\boldsymbol{m}'\chi'}$. We know that in the Lorentz group we have

$$(V_{\boldsymbol{m}',\chi'}, \mathbb{1})(V_{\boldsymbol{m},\chi}, \mathbb{1}) = (V_{\boldsymbol{m}'',\chi''}, R_\varphi). \tag{3}$$

The problem of finding $\boldsymbol{m}''$, χ'' and R_φ in terms of $\boldsymbol{m}'$, χ', $\boldsymbol{m}$, χ can be translated to a problem of composition of isometries in hyperbolic space. Furthermore, as a consequence of the commutation relations in this group, $\boldsymbol{m}''$ is spanned by $\boldsymbol{m}'$, $\boldsymbol{m}$ and the axis of R is orthogonal to this plane. So we may well restrict ourselves to the hyperbolic plane.

In the following, we simply write (3) by

$$(\boldsymbol{m}'\chi', \mathbb{1})(\boldsymbol{m}\chi, \mathbb{1}) = (\boldsymbol{m}''\chi'', \varphi). \tag{4}$$

3. Lorentz turns

Once in the hyperbolic plane, a good method for the composition of direct isometries is to express them as products of reflections. A translation of magnitude χ along a line Γ, $T_{\Gamma,\chi}$, is just the product of two reflections in any two lines orthogonal to Γ, at a distance $\frac{1}{2}\chi$, and a rotation of angle φ, around 0, is just the product of two reflections in any two lines passing through 0, and at an angle $\frac{1}{2}\varphi$. With relation to translations any pair of points P and P' on Γ, at a distance $\frac{1}{2}\chi$, can be chosen as intersections of γ and γ' (orthogonal lines to Γ through P and P') with Γ, so that it is natural to associate to $T_{\Gamma,\chi}$ an oriented segment of length $\frac{1}{2}\chi$ on Γ, but otherwise free to slide on Γ. As we shall see, this is analogous to Hamilton's turns, and will be called a Lorentz turn. The fact that these objects reproduce, by means of a 'parallelogram-like' law, the composition of translations is an immediate consequence of a result in absolute geometry called Donkin's theorem.

Theorem. The product of the three translations along the oriented sides of a triangle, and of magnitude twice the lengths of its sides, is equal to the identity.

The proof of this result is given e.g. in Coxeter (1969) and consists simply of decomposing each translation $T_{\Gamma,\chi}$ as the product of two 'half-turns' or rotations of angle π around two points P and P' on Γ, at a distance $\frac{1}{2}\chi$ which are the vertices of the triangle. Notice that a very similar result holds for rotations in ordinary space (Biedenharn and Louck 1981, p 192), and that in Euclidean geometry the result is true for translations even if we replace 'twice' by any other proportionality constant.

A minor difference with Hamilton's turns is that, whereas all rotations have an associated turn, not all hyperbolic isometries, but only translations have an associated Lorentz turn. This is irrelevant for the following discussion.

This result gives us the following construction for the product of two translations $T_{\Gamma,\chi}$ and $T_{\Gamma',\chi'}$, along intersecting lines Γ and Γ'. Take the turns $\mathscr{T}$ and $\mathscr{T}'$ associated with the respective translations, and slide them along Γ and Γ' until they are 'head to tail'. Afterwards the turn determined by the free tail and head, $\mathscr{T}''$, is the turn associated to the product $T_{\Gamma',\chi'}T_{\Gamma,\chi}$, which is thus a translation along the line Γ'' of magnitude $2d(P, Q)$. This construction is shown in figure 1.

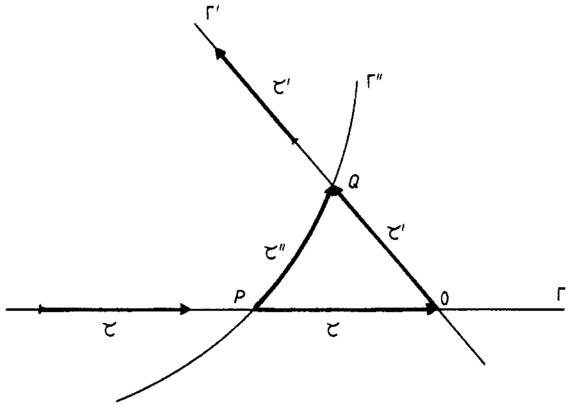

Figure 1. Evaluation of the product of two hyperbolic translations $T_{\Gamma,\chi}$ and $T_{\Gamma',\chi'}$ by means of the associated turns $\mathscr{T}$ and $\mathscr{T}'$. The turns $\mathscr{T}$ and $\mathscr{T}'$ (of hyperbolic lengths $\frac{1}{2}\chi$ and $\frac{1}{2}\chi'$) are slid on Γ and Γ' until they are tail to head at 0. Then the free 'tail' P and 'head' Q determine the turn $\mathscr{T}''$ associated to the product $T_{\Gamma',\chi'} \circ T_{\Gamma,\chi}$ which is thus identified as a translation along the straight line Γ'' of parameter $\chi'' = 2d(P, Q)$.

Let us take a specific viewpoint and seek the relevance of the preceding construction to our problem. We choose the Poincaré disc model (Pedoe 1970) of the hyperbolic plane, and consider the lines through its centre 0 (diameters of the circle from a Euclidean viewpoint). Any direct isometry is a product of a rotation around 0 and a translation along a line through 0. Furthermore, the conformal nature of the model implies that the (hyperbolic) angle between two such lines coincides with the Euclidean angle, so that such a line can be characterised (besides the point 0) by the (Euclidean) unit vector of the diameter $\boldsymbol{m}$; the hyperbolic angle between two lines is equal to the ordinary angle between $\boldsymbol{m}$ and $\boldsymbol{n}$.

Let us return to our original problem. Evaluate the product $(\boldsymbol{m}'\chi', \mathbb{1})(\boldsymbol{m}\chi, \mathbb{1})$. By means of the isomorphism (1), this is equivalent to finding the product of the translations $T_{\boldsymbol{m}',\chi'}T_{\boldsymbol{m},\chi}$. Using the Lorentz turn construction this product is itself a translation of magnitude ψ along the line Γ, as shown in figure 2. This translation can only be decomposed into a product,

$$T_{\Gamma,\psi} = T_{\boldsymbol{m}'',\chi''}R_{0,\varphi}$$

which by using (4) gives back

$$(\boldsymbol{m}'\chi', \mathbb{1})(\boldsymbol{m}\chi, \mathbb{1}) = (\boldsymbol{m}''\chi'', \varphi).$$

The problem is then solved once we are able to extract from the pair Γ, ψ the elements $\boldsymbol{m}''$, χ'' and φ. A very simple construction allows this to be done, if we use the decomposition of isometries into product of reflections. Now, take the perpendicular Δ to γ' through 0, and the perpendicular δ to Δ from 0. As $(I_\delta)^2 = \mathbb{1}$ we have

$$T_{\Gamma,\psi} = I_{\gamma'}I_\gamma = I_{\gamma'}I_\delta I_\delta I_\gamma = T_{\Delta,\chi''}R_\varphi$$

where $\frac{1}{2}\chi''$ is the distance between 0 and γ', and $\frac{1}{2}\varphi$ is the angle between γ and δ. The line Δ, which contains 0, can be characterised by its own vector $\boldsymbol{m}''$. Hence this diagram gives us $\boldsymbol{m}''$, χ'' and φ at the same time.

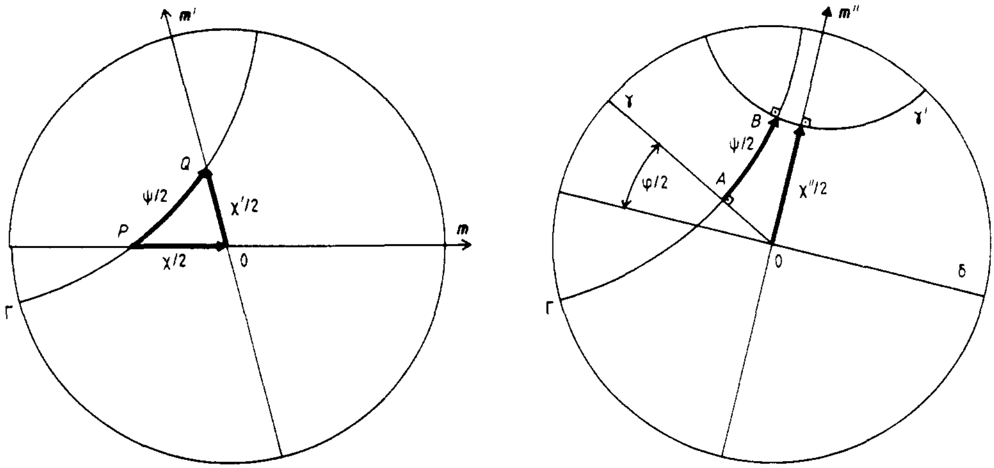

Figure 2. Lorentz's turn construction in the Poincaré disc model corresponding to a product of two pure inertial transformations $(\boldsymbol{m}'\chi', \mathbb{1})(\boldsymbol{m}\chi, \mathbb{1}) = (\boldsymbol{m}''\chi'', R_\varphi)$. (*a*) Product of the translations $T_{0\boldsymbol{m}',\chi'} \circ T_{0\boldsymbol{m},\chi} = T_{\Gamma,\psi}$. (*b*) Decomposition of $T_{\Gamma,\psi}$ as a product $T_{\Gamma,\psi} = T_{0\boldsymbol{m}'',\chi''} \circ R_\varphi$. Here $d(P, Q) = d(A, B)$ and ⌐ denotes orthogonal lines.

In this way we obtain the following prescription (Juárez 1981) for the product $(\boldsymbol{m}''\chi'', R_\varphi)$ of two pure inertial transformations $(\boldsymbol{m}\chi, \mathbb{1})$ and $(\boldsymbol{m}'\chi', \mathbb{1})$.

(i) Draw the corresponding turns as in figure 2, and obtain the line Γ and the arc PQ.

(ii) Slide the arc PQ keeping its length unaltered until its tail is at the foot of the perpendicular γ to Γ (see figure 2) passing through 0.

(iii) Then draw γ', perpendicular to Γ cutting at the head of the arc. Now $\boldsymbol{m}''$ is the direction of the perpendicular to γ', passing through 0, and the rapidity χ'' is twice the distance between 0 and γ'.

(iv) The angle φ is exactly twice the angle between the lines γ and δ.

A number of remarks must be made.

(*a*) As the model is conformal at 0 and the lines through 0 are also Euclidean lines, all queries about angles can be read immediately from the diagram (with 'Euclidean glasses' so to speak).

(*b*) As the model is not, of course, isometric, the Euclidean length of the turn $\mathscr{T}$ varies as it slides.

This makes the precise determination of the point B somewhat difficult; the positions of P and Q being known. Later we shall develop a slide rule which helps in an exact construction. But, at the last level of the construction, all turns meet at 0, and hence only distances from 0 to other points are relevant. The relation between the hyperbolic distance d and the Euclidean one r, between 0 and some generic point X in the Poincaré model, is known:

$$r = \tanh(\tfrac{1}{2}d). \tag{5a}$$

Thus the rapidity of the motion with turn $0X$ is $\chi = 2d$, and the relation between r and the velocity V is given by

$$r = \tanh(\tfrac{1}{4}\tanh^{-1} V) \tag{6a}$$

or conversely

$$V = \tanh(4 \tanh^{-1} r). \tag{7a}$$

This is the 'universal' law of transcription velocity–Euclidean-length of the turns at 0, in the Poincaré model. Near the point 0 we have $r \simeq \frac{1}{4}V$, and we recover the ordinary parallelogram law.

(c) The Klein model which is conformal only at 0, and whose lines are (Euclidean) straight lines, is also very adequate for our purposes (Artzy 1974, p 174). A small drawback is that this model is only conformal at 0. The relation (5) for the Klein model is also known

$$r = \tanh d \tag{5b}$$

which now gives us the universal relations

$$r = \tanh(\tfrac{1}{2}\tanh^{-1} V) \tag{6b}$$

$$V = \tanh(2 \tanh^{-1} r). \tag{7b}$$

The formula (6b) behaves as $r = \frac{1}{2}V$ for small velocities. The Klein model is very suitable under some circumstances (see figure 3).

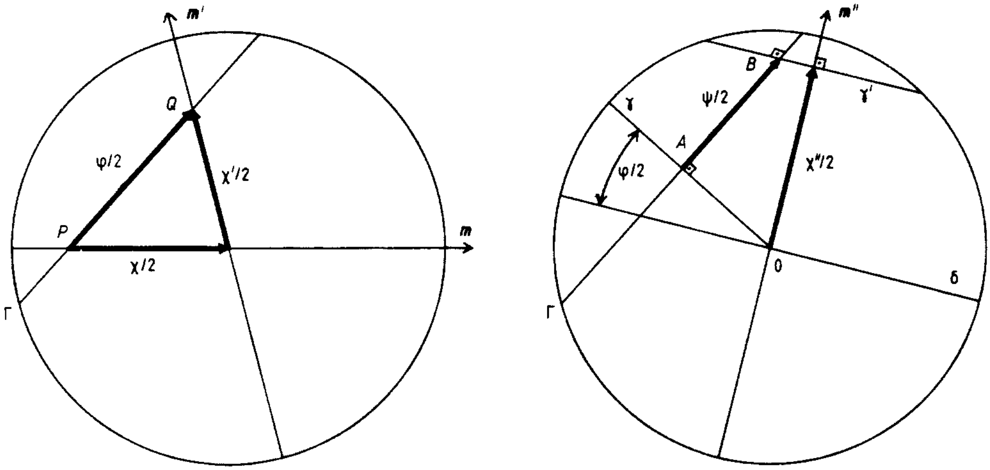

Figure 3. Lorentz's turns construction in the Klein model. Notation as in figure 2.

(d) Both models (Poincaré and Klein) are simply related to those based on $\Omega^+_{r=1}$. In fact the Klein model is obtained by stereographic projection of $\Omega^+_{r=1}$ in the plane $t = 1$, from the origin of Minkowski space, and the Poincaré model is obtained from the Klein model in the usual way. To be more specific, if one parametrises $\Omega^+_{r=1}$ through the point $(x, y) \in \mathbb{R}^2$ and uses 'polar' coordinates (r, φ), and z denotes the complex affix of the generic point of the unit circle supporting both models, the relation is

Klein model $\quad z = e^{i\varphi} \tanh(\frac{1}{2}r)$

Poincaré model $\quad z = e^{i\varphi} \tanh(\frac{1}{2}\sinh^{-1} r).$

(*e*) The diagrams are to be understood in a 'neutral' space, where space vectors (as $\boldsymbol{m}, \boldsymbol{m}', \boldsymbol{m}''$) measured in different frames are mixed. Of course, the same trivial observation holds for the Euclidean diagrams used, e.g. for classic aberration.

(*f*) One may even envisage another model of the hyperbolic plane in the unit circle of the Euclidean plane, with the relations (5) and (6) substituted by $r = \tanh 2d$, so that the universal relation $V \leftrightarrow r$ is then simply $r = V$. This model would be very convenient because the Euclidean length of the turn at 0 would give the velocity of the corresponding pure inertial transformation directly. But this would cause a complication in all other properties of the model.

4. Applications

The geometrical construction developed in the previous section can be used in all physical problems involving the composition of velocities, from simple aberration, both of massive particles and of light, to the appearance of bodies in rapid motion and the Thomas precession. It is not our intention here to discuss all the possible applications exhaustively, so that we restrict ourselves to some indications.

First, only in order to establish the notation, we shall say that a reference system, K', is obtained from another, K, by means of a Lorentz transformation $(\boldsymbol{V}, R)$ and we shall write

$$K \xrightarrow{(\boldsymbol{V},R)} K'$$

if any fixed point in K moves respectively to K' with velocity $\boldsymbol{V}$ (as measured relative to K'), and if the axis of K can be obtained from those of K' by means of the rotation R (as measured also relative to K'). Of course the rapidity $\boldsymbol{\chi}$ could be used instead of $\boldsymbol{V}$.

4.1. *Aberration*

Let K, K' be two inertial frames, and $\boldsymbol{V}$ the velocity of K' as measured in K, and let a particle move in K with velocity $\boldsymbol{v}$. What is the velocity $\boldsymbol{v}'$ of the particle relative to K'? We introduce another inertial frame K'' moving with velocity $\boldsymbol{V}$ relative to K, but with the same orientation of spatial axes. It is clear that $\boldsymbol{v}'$ is determined by the diagram,

$$\begin{tikzcd} K'' \arrow[r, "(\boldsymbol{v}'R')"] \arrow[rr, bend right, "(\boldsymbol{v},\mathbb{1})"'] & K' \arrow[r, "(\boldsymbol{V},\mathbb{1})"] & K \end{tikzcd}$$

and hence $\boldsymbol{v}'$ is the 'velocity-like part' of the product

$$(-\boldsymbol{V}, \mathbb{1})(\boldsymbol{v}, \mathbb{1}).$$

The diagram in the Poincaré model is shown in figure 4. For simplicity, the turn associated with the pure inertial transformation of velocity $\boldsymbol{V}$ is simply labelled as $\boldsymbol{V}$ in the diagram. The direction of $\boldsymbol{V}$ can be read directly, whereas the modulus must be read using the formula (7*a*).

Ordinarily, the discussion of this phenomenon makes use of the angles θ, θ' of the motion of the particle with $\boldsymbol{V}$ as measured in K and K'. These angles are also indicated in our diagram.

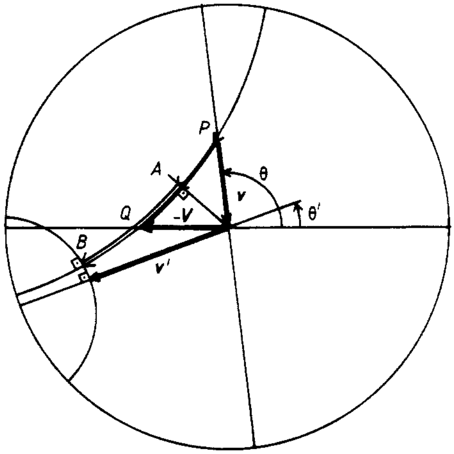

Figure 4. Diagram for aberration evaluated by means of Lorentz's turns in the Poincaré model. Turns at 0 are simpiy labelled by their speeds.

4.2. *Thomas precession*

Let an electron describe a plane orbit around a proton. If K is an inertial frame where the proton is at rest, the velocity of the electron, as measured in K, varies with time. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be the instantaneous velocities of the electron at some instants t_1 and t_2, and introduce two systems K' and K'', both moving respectively to K at speeds $\boldsymbol{V}$ and $\boldsymbol{W}$, otherwise both with the same orientation of spatial axes as K. Here the interesting unknown factor is the angle of the rotation R_φ, which must be applied to the axes of K' to make them coincide with the axes of K''. Now the scheme is

$$\begin{tikzcd} K' \arrow[r, "{(v', R)}"] \arrow[rr, bend right, "{(v,\, \mathbb{1})}"'] & K'' \arrow[r, "{(w, \mathbb{1})}"] & K \end{tikzcd}$$

and hence R_φ is the 'rotation part' of the product $(-\boldsymbol{W}, \mathbb{1})(\boldsymbol{V}, \mathbb{1})$. If the electron describes a circular orbit of angular velocity ω, then the angle between $\boldsymbol{V}$ and $\boldsymbol{W}$, at times t and $t+\mathrm{d}t$, is $\omega\, \mathrm{d}t$. The construction which gives the angle $\mathrm{d}\varphi$ is given in the diagram of figure 5; this time in the Klein model.

When $|\boldsymbol{v}| \ll 1$, we can even obtain a numerical expression with the most elementary methods from the diagram. The Euclidean length of the turn $(-\boldsymbol{w}, \mathbb{1})(\boldsymbol{v}, \mathbb{1})$ is nearly $\frac{1}{2}v\omega\, \mathrm{d}t$, because $|\boldsymbol{v}| \ll 1$. This Euclidean length is unaltered in first order when the turn is slid until its tail touches Q. Elementary geometry then says (with the same approximation) that $PQ \simeq 2/v$. Hence $\frac{1}{2}\mathrm{d}\varphi \simeq \frac{1}{4}v^2\omega\, \mathrm{d}t$ and $\Omega = \mathrm{d}\varphi/\mathrm{d}t \simeq \frac{1}{2}v^2\omega$, which is of course, the low-velocity limit of the Thomas frequency. The same construction, without any approximation, would give the exact value of Ω.

5. The case of light

When the motion of light is considered, the whole discussion seems to be irrelevant at first glance. But both of our models allow a 'limit' $\chi \to \infty$ (or $v \to 1$) to be made in a smooth way. Then the turns, so to speak, are of infinite hyperbolic length, and its

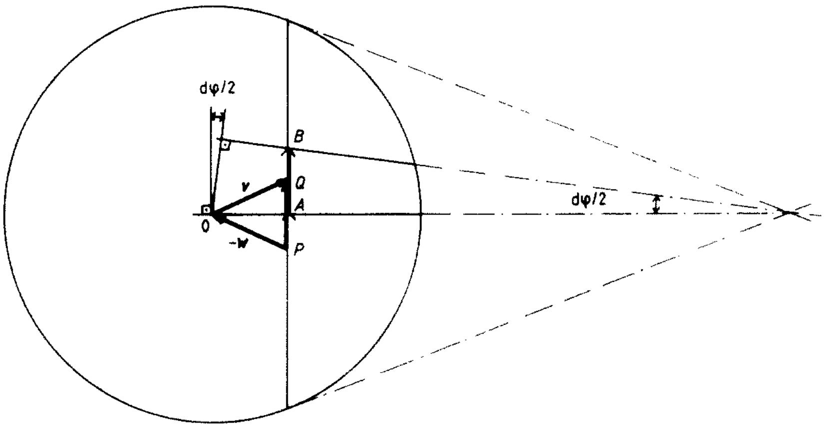

Figure 5. Diagram for the Thomas precession in Klein's model.

head or tail must lie on the boundary of the circle (which, of course, is outside the hyperbolic plane). Thus, in fact our construction is, at the same time, valid and simplified. The aberration diagram for the light in the Klein model is surprisingly simple, as is shown in figure 6(a). This construction is free of almost all the complications of the preceding ones, and illustrates very well the most rapid (when compared with classical predictions) change in the apparent positions of stars when the sky is seen from a space ship. The same diagram can be successfully used in a qualitative discussion of the photographic appearance of an object moving at great speeds (see e.g. Hagedorn 1964, § 5.5, Hickey 1979 and references therein). We leave both applications as a useful exercise to the reader.

The case of light gives rise to yet a further curious result. The transition of a reference frame K to another K', besides the aberration, causes a Doppler effect. Whereas a transversal motion (at any velocity) causes no Doppler effect according to classical mechanics, it produces a redshift in relativity. It is well known that the

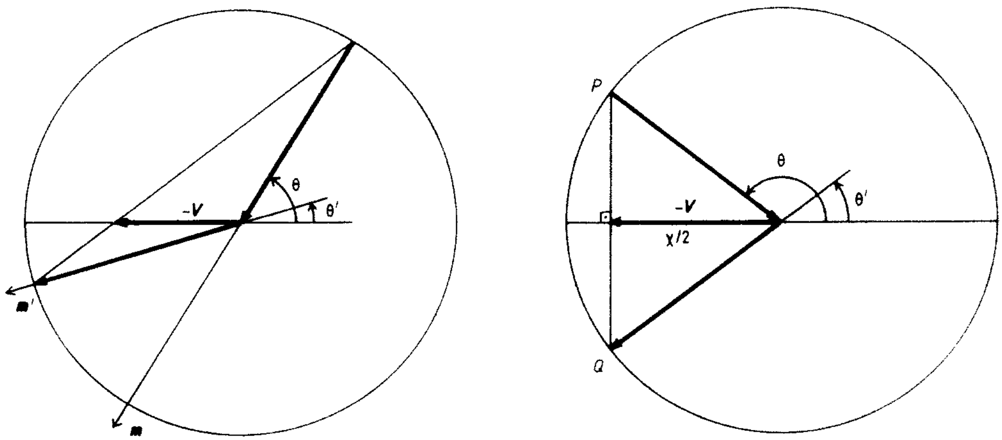

Figure 6. (a) Diagram for light aberration in Klein's model. (b) Interpretation of the angle of parallelism $\theta'(\frac{1}{2}\chi)$ in terms of relativistic kinematics.

Doppler factor D can be expressed in terms of θ and θ' as

$$D = \sin\theta/\sin\theta'$$

(of course it depends on θ and V, through θ').

When we have $D = 1$, this means θ and θ' are supplementary, so that the line PQ is orthogonal to the turn $\mathscr{T}(\boldsymbol{V})$; it is clear from the diagram, that $\theta' = \pi - \theta$ is the angle of parallelism of the length $\frac{1}{2}\chi$, of the turn associated to $\boldsymbol{V}$ (see figure 6(b)). This gives an unexpected interpretation of the angle of parallelism in terms of relativistic kinematics; that light emitted by a source moving at a velocity $\boldsymbol{V}$, at an angle equal to the hyperbolic angle of parallelism of the length $\frac{1}{2}\chi$, with the line of observation, would be received unshifted by us. This can be checked by direct calculation using the standard expressions for the Doppler shift, instead of the geometric reasoning used before. Anyway, the result is so neat that it would be perhaps interesting to have a simpler, physical argument leading to it. It is very amusing to note that in principle (only in principle, of course!) hyperbolic geometry could have been discovered experimentally in this or any similar way. Of course, in the classical limit, no Doppler effect appears for transversal motion, or in geometrical terms, in Euclidean geometry the angle of parallelism of whatever distance is always equal to $\frac{1}{2}\pi$.

Acknowledgments

One of the authors (MS) would like to express his gratefulness to Professor Fernando Vallepuga from whom he first learned non-Euclidean geometry.

Appendix. A slide rule for Lorentz's turns

We briefly describe a slide rule which permits us to carry out the process of composition of Lorentz's turns described in the text. It is based on Poincaré's model. A similar idea for the case of rotations using Hamilton's turns has been developed by Harter and dos Santos (1978). From a practical viewpoint it is possible to obtain numerical results with remarkable accuracy.

The slide rule has two transparent turning scales A and B, and a lower one, C, which is fixed and contains only an angular scale (see figure 7). The scales A and B are fastened so that their centres stay precisely over the centre of C, 0. On A and B there is a fixed (relative to C) transparent sheet, D. We can draw the Lorentz turns on it.

The scale A shows a pencil of ultraparallels, i.e. all straight lines perpendicular to a straight line passing through 0; on this line (the diameter d) there is a scale of velocities obtained through formula (7a). The pencil of coaxial curves equidistant to d (orthogonal to the former pencil of ultraparallels) is also shown. On the periphery of the circle there is a half-angle angular scale.

The scale B shows a pencil of ultraparallels orthogonal to a ray s through 0. The pencil of coaxial equidistant curves to this ray is also displayed. On the diameter d' there is a scale of hyperbolic distances between the ray and the corresponding equidistant curve.

We shall illustrate the use of this rule in a concrete example (figure 8). Let $\boldsymbol{U}$ be along the positive x axis with $|\boldsymbol{U}| = 0.9$ and $\boldsymbol{V}$ at an angle of 45°, $|\boldsymbol{V}| = 0.7$. To evaluate with the above rule the velocity $\boldsymbol{W}$ and the angle φ in the product

$$(\boldsymbol{U}, \mathbb{1})(\boldsymbol{V}, \mathbb{1}) = (\boldsymbol{W}, R_\varphi)$$

we must follow the procedure given below.

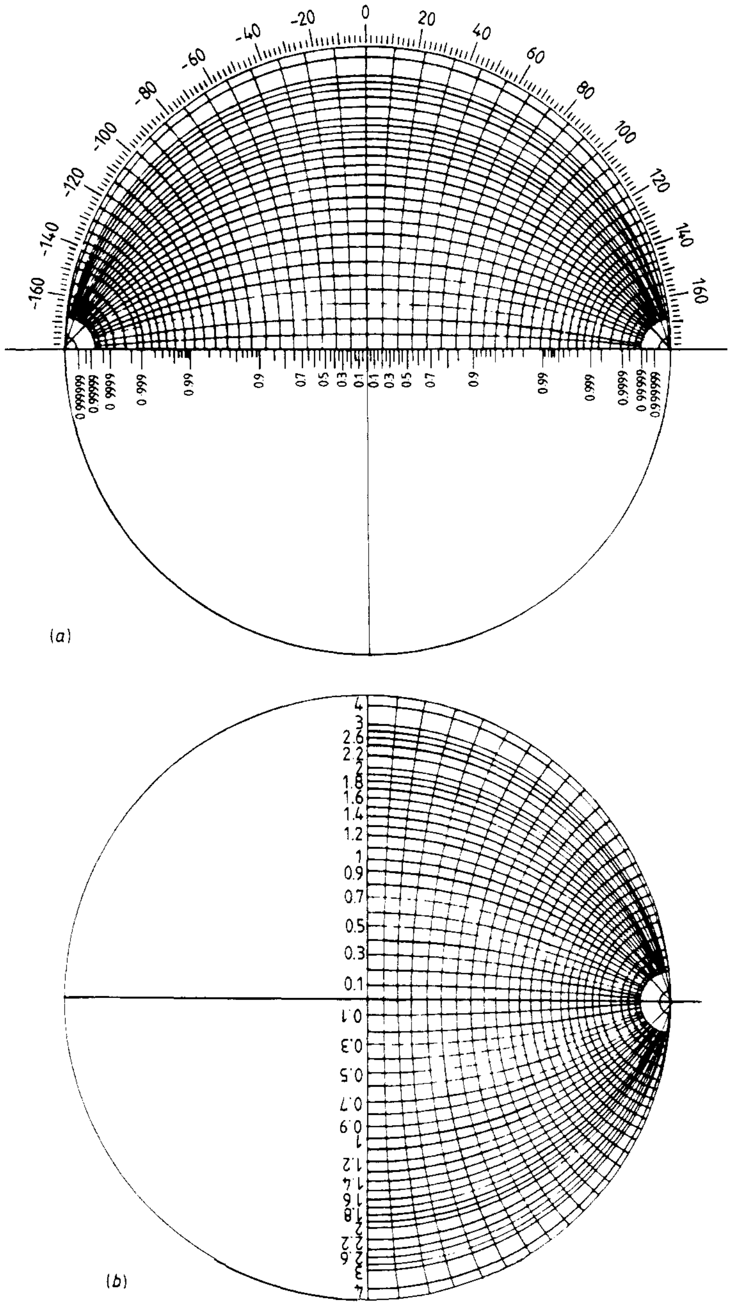

Figure 7. Slide rule for Lorentz's turns; (*a*) scale A, (*b*) scale B.

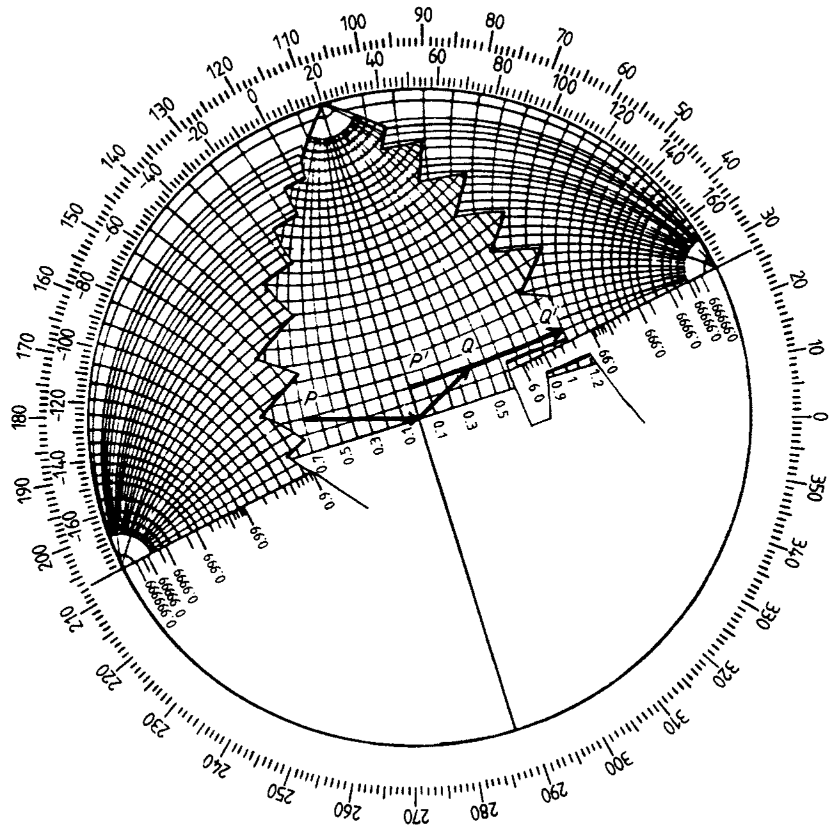

Figure 8. An example of the use of the slide rule.

(i) Draw on the fixed sheet D the turns corresponding to $\boldsymbol{U}$ and $\boldsymbol{V}$ correctly placed. This is easily done using the scale of velocities and the indicator of d over the scale C. We obtain points $P \equiv$ tail of $\boldsymbol{V}$ and $Q \equiv$ head of $\boldsymbol{U}$.

(ii) By turning B, look for the straight line r passing through P and Q. The length of the product turn $\mathscr{T}$ is read using the scale on B, by means of the equidistant lines through P and Q. Here $d(P, Q) = 0.4 + 0.7 = 1.1$.

(iii) The turn $\mathscr{T}$ must be slid until its tail lies on the foot P' of the perpendicular to r from 0. The head of $\mathscr{T}$ is now on the intersection Q' of r and the equidistant curve 1.1.

(iv) Using A look for the perpendicular r' to r through Q'. The intersection point of r' with d gives the modulus of $\boldsymbol{W}$ on the scale of velocities and the direction of $\boldsymbol{W}$ is that of d, measured by the indicator of d over the angular scale C. In our case, $|\boldsymbol{W}| \approx 0.976$ and $\boldsymbol{W}$ is directed at $\approx 27°$. These values are very close to the exact ones obtained analytically, $|\boldsymbol{W}| = 0.9765$ and $26.21°$.

(v) The angle φ is directly read over the half-angle angular scale on A by means of the indicator of the ray s of B. Here $\varphi \approx 20°$.

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